A Test of Phillips' Hypothesis for Eddy Viscosity in Pipe Flow

LIONEL V. BALDWIN and ROBERT D. HABERSTROH

Colorado State University, Fort Collins, Colorado

In a recent paper, O. M. Phillips (*1*) proposed a mechanism for the manner in which turbulent components support Reynolds stress in turbulent shear flow. Phillips' model is a generalization of Miles' mechanism for wind generated water waves, in that each turbulent component is assumed to interact with the mean flow to produce an increment of Reynolds stress at the matched layer of that particular component. The derivation is rather involved but leads to a simple relation between measurable turbulence statistical properties and eddy viscosity. The eddy viscosity derived by Phillips is not the customary formulation of Boussinesq, but it is easily related to the latter. Specificaüy, for a fully developed pipe flow, Phillips gives

$$\frac{d}{dr}(r\tau_{rx}) = \mu_e \frac{d}{dr}\left(r\frac{dU}{dr}\right) \tag{1}$$

where

$$\tau_{rx} = -\rho\ \overline{uw}$$

whereas the classical formulation is (*2*, p. 23),

$$\tau_{rx} = \mu_e^* \frac{dU}{dr} \tag{2}$$

In other words, Phillips' mechanism leads to an eddy viscosity μ_e which is a proportionality constant between the stress gradient and the second derivative of the mean velocity rather than the familiar form [Equation (2)]. Furthermore, the mechanism relates μ_e to measurable physical properties of the turbulence; thus

$$\mu_e = A\ \rho\ \overline{w^2}\ \Theta \tag{3}$$

where A is a number less than π and Θ the convected integral time scale of the lateral fluctuation velocity which has a mean square magnitude, $\overline{w^2}$. Phillips tested the analysis with experimental anemometer data obtained in the near field mixing region of an air jet (*3*) and he concluded that these data were consistent with the analytical prediction. For the jet flow mixing region, $A = 0.24$, but Phillips states that, depending on the shape of the turbulent eddy, the precise value may vary somewhat from one turbulent shear flow to another.

Eddy viscosity models are widely used, and although there have recently been numerous advances in formulating rather general functions (*4*), heuristic dimensional arguments are usually the sole basis of the formulation. Phillips' proposal is a welcome exception which should be critically tested in a variety of turbulent shear flows. This paper summarizes the results of one such test in the core of fully developed pipe flow.

The shear stress gradient is constant in a fully developed pipe flow and it is easily related to the wall pressure drop down the tube. A summary of measured wall shear stress data for smooth tubes, as well as the ratio of bulk to centerline velocity, is given elsewhere (*5*) in the form of a review of the semiempirical velocity profile proposed by Pai (*2*, p. 42).This velocity profile is the most accurate available in the core region of pipe flow. The eddy viscosity of Equation (1) may be derived using the Pai profile to be

$$\frac{\mu_e(\eta)}{\mu} = \frac{s}{\dfrac{n-s}{n-1} + n\dfrac{s-1}{n-1}\eta^{2n-2}} - 1 - \frac{ns\,(s-1)\ \eta^{2n-2}}{\left[\dfrac{n-s}{n-1} + n\dfrac{s-1}{n-1}\eta^{2n-2}\right]\left[\dfrac{n-s}{n-1} + n^2\dfrac{s-1}{n-1}\eta^{2n-2}\right]} \tag{4}$$

Here n and s are empirical constants which are functions only of the Reynolds number; numerical values for these constants in graphical and tabular form are given in (*5*). At the pipe centerline, the dimensionless radius, $\eta = \dfrac{r}{a} = 0$, and the Phillips eddy viscosity, μ_e, becomes identical to the classical version, μ_e^*:

$$\frac{\mu_e(0)}{\mu} = \frac{n(s-1)}{n-s} \tag{5}$$

Figure 1 is a plot of Equation (5) prepared from the n and s values of reference 5. Although not shown in the plot, $\mu_e(0) = 0$ at $N_{Re} \lesssim 2{,}100$ because s is unity by definition in laminar pipe flow.

The statistical properties of turbulence in pipe flow necessary to test Phillips' model [Equation (3)] have been published (*6*, *7*). The Eulerian space-time correlation of the axial velocity fluctuations in the apparent convective frame (*7*) were fit with exponential curves for the reported four mean flow velocities of air flowing in an 8 in. pipe (see Table 1). The convective integral scales, L_T', (*3*) were then read as the values of τ where the peak correlations dropped to the value of $1/e$. This procedure is identical to that followed by Phillips (*1*) in interpreting the jet mixing region data of Davies (*3*) in his original computation of A. The lateral intensities of turbulence, $\overline{w^2}$, were reported for the same experimental

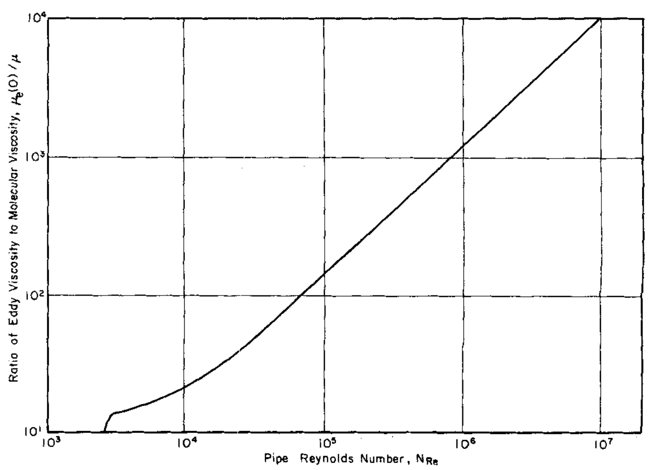

Fig. 1. Eddy viscosity at the centerline of fully developed pipe flow.

TABLE 1. CALCULATION OF PHILLIPS' EDDY VISCOSITY CONSTANT

$U_{℄}$, ft./sec. (6)	$\overline{U}$, ft./sec. (Calc.)*	$N_{Re} = \frac{\overline{U}2a}{\nu}$ (Calc.)†	L_T' sec. (7)	$\overline{w^2}$, sq.ft./sq.sec. (6)‡	n (5)	s (5)	$\frac{\mu_e(0)}{\mu}$ (Calc.)	A (Calc.)
72.6	61.4	255,000	0.0352	3.62	142	100	338	0.426
106.0	90.0	372,000	0.0305	7.71	194	137	466	0.319
135.0	115.0	476,000	0.0298	12.5	237	170	600	0.260
160.0	136.0	573,000	0.0208	17.6	277	200	720	0.317

$A = 0.33$ ave.

* $\overline{U} = \frac{n+s}{2(n+1)} U_{℄}$ (5).

† $2a = 8'' = 0.667$ ft; $\nu = 1.61 \times 10^{-4}$ sq.ft./sec.

‡ Calc. from $\sqrt{\overline{w^2}} = 0.0262\ U_{℄}$.

conditions (6) in the following form:

$$\sqrt{\overline{u^2}} = 0.035\ U_{℄}$$

and

$$\sqrt{\overline{w^2}} \simeq 0.75 \sqrt{\overline{u^2}}$$

Table 1 summarizes the turbulent properties, L_T' [which Phillips used for Θ in (1)] and $\overline{w^2}$, as well as the value of A computed from

$$A = \frac{\mu_e(0)}{\rho \overline{w^2}\ L_T'} \tag{3b}$$

In pipe flow the values of A range from 0.43 to 0.26 with no systematic Reynolds number trend. The average value of A for pipe flow is 0.33 which is only 30% larger than the value inferred by Phillips from jet data. Although more definitive experimental tests of Phillips model need to be designed, the pipe flow results are consistent in the same sense as the original experimental test employing jet data.

It is worth noting in closing that a direct measurement of the turbulent diffusivity of heat, α_t, for the core of pipe flow was reported [(6) (Table 1)] for these identical flow conditions. By using the kinematic eddy viscosity, $\nu_e(0)$, calculated from Equation (5) divided by the air density, an eddy Prandtl number, N_{Prt}, at the pipe centerline may be computed. Table 2 shows N_{Prt} is essentially unity, which lends direct support to the classical version of the Reynolds analogy. In the same vein, Phillips' eddy viscosity formulation [Equation (3)] bears a striking resemblance to the analogous eddy diffusivity of heat or mass which would be calculated from Taylor's theory of diffusion by continuous movements using the Eulerian-to-Lagrangian approximations proposed by others (6, 7).

TABLE 2. MEASURED TURBULENT PRANDTL NUMBERS

N_{Re}	ν_e, sq.ft./sec.	α_t* sq.ft./sec.	$N_{Prt} = \frac{\nu_e}{\alpha_t}$
255,000	0.0544	0.054	1.00
372,000	0.0752	0.072	1.04
476,000	0.0966	0.088	1.10
573,000	0.116	0.12	0.967

Avg. $N_{Prt} = 1.02$

* Taken from Table 1, reference 6.

NOTATION

a = pipe radius, ft.
A = dimensionless constant, Equation (3)
e = Naperian logarithm base, 2.718 ...
L_T' = convective integral scale of axial turbulent velocity from space-time, data, sec.
n = dimensionless constant, Equation (4)
N_{Prt} = $\frac{\nu_e}{\alpha_t}$, turbulent Prandtl number
N_{Re} = $\frac{\overline{U}2a}{\nu}$, Reynolds number
r = radial coordinate, ft.
s = dimensionless constant, Equation (4)
$\overline{U}$ = bulk mean flow velocity, ft./sec.
$U_{℄}$ = centerline velocity, ft./sec.
$\overline{uw}$ = Reynolds shear stress component, sq.ft./sq.sec.
$\overline{u^2}$ = mean square of axial turbulent velocity, sq.ft./sq.sec.
$\overline{w^2}$ = mean square of radial turbulent velocity, sq.ft./sq.sec.
Θ = convective integral scale of radial turbulent velocity from space-time data, sec.
η = $\frac{r}{a}$, dimensionless pipe radius
μ = molecular viscosity, $lb._F$-sec./sq.ft.
μ_e = Phillips' eddy viscosity, Equation (1), $lb._F$-sec./sq.ft.
μ_e* = Boussinesq's eddy viscosity, Equation (2), $lb._F$-sec./sq.ft.
$\nu_e \equiv \frac{\mu_e}{\rho}$ = kinematic eddy viscosity, sq.ft./sec.
ρ = fluid density, $lb._F$-sec./ft.^{4}
τ_{rx} = turbulent shear stress, $lb._F$/sq.ft.

LITERATURE CITED

1. Phillips, O. M., *J. Fluid Mech.*, **27**, 131 (1967).
2. Pai, S. I., "Viscous Flow Theory," Vol. II, "Turbulent Flow," Van Nostrand Co., (1967).
3. Davies, P. A., et al., *J. Fluid Mech.*, **15**, 337 (1963).
4. Mellor, G. L., *AIAA Journal*, **5**, No. 9, 1570 (1967).
5. Haberstroh, R. D., and L. V. Baldwin, *J. Heat Transfer*, **90**, 191 (1968).
6. Baldwin, L. V., and T. J. Walsh, *AIChE J.*, **7**, 53 (1961).
7. ———, and W. R. Mickelsen, *Trans. Am. Soc. Chem. Eng.*, **128**, No. 1, 1595 (1963).